Lecture 04
12.4/12.5 The cross product and lines in space

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January 23, 2019

Office Hours

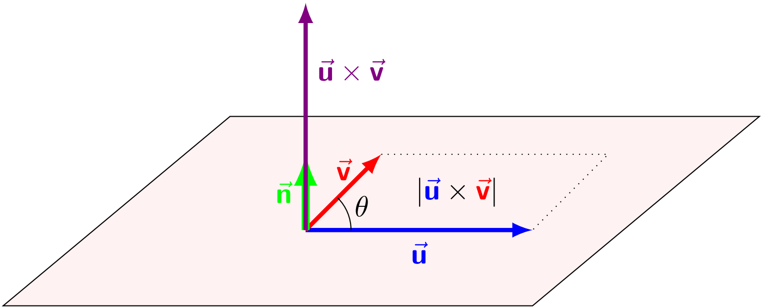
MW 2:40-3:40
T 9:30-10:30
R 12:30-1:30
F 8:30-9:30

Last Class

Definition

The cross product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, denoted $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, is the vector

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \sin(\theta))\vec{\mathbf{n}}.$$



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We also have the property that $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$, i.e., that the cross product distributes across vector addition. (This is not obvious!)

Cross Product Component Formula

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}}$$
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Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then we have

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}} + u_3\vec{\mathbf{k}}) \times (v_1\vec{\mathbf{i}} + v_2\vec{\mathbf{j}} + v_3\vec{\mathbf{k}})$$

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$$\begin{aligned} &= u_1 v_1(\vec{\mathbf{i}} \times \vec{\mathbf{i}}) + u_1 v_2(\vec{\mathbf{i}} \times \vec{\mathbf{j}}) + u_1 v_3(\vec{\mathbf{i}} \times \vec{\mathbf{k}}) \\ &+ u_2 v_1(\vec{\mathbf{j}} \times \vec{\mathbf{i}}) + u_2 v_2(\vec{\mathbf{j}} \times \vec{\mathbf{j}}) + u_2 v_3(\vec{\mathbf{j}} \times \vec{\mathbf{k}}) \\ &+ u_3 v_1(\vec{\mathbf{k}} \times \vec{\mathbf{i}}) + u_3 v_2(\vec{\mathbf{k}} \times \vec{\mathbf{j}}) + u_3 v_3(\vec{\mathbf{k}} \times \vec{\mathbf{k}}) \end{aligned}$$

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$$\begin{aligned} = \quad 0 \quad &+ u_1 v_2(\vec{\mathbf{i}} \times \vec{\mathbf{j}}) + u_1 v_3(\vec{\mathbf{i}} \times \vec{\mathbf{k}}) \\ + u_2 v_1(\vec{\mathbf{j}} \times \vec{\mathbf{i}}) &+ \quad 0 \quad + u_2 v_3(\vec{\mathbf{j}} \times \vec{\mathbf{k}}) \\ + u_3 v_1(\vec{\mathbf{k}} \times \vec{\mathbf{i}}) &+ u_3 v_2(\vec{\mathbf{k}} \times \vec{\mathbf{j}}) + \quad 0 \end{aligned}$$

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The components of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ can be recognized as determinants.

2x2 Determinants

Given a 2x2 matrix, we can calculate its determinant as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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3x3 Determinants

Given a 3x3 matrix, we can calculate its determinant as follows (beware of the minus before a_2):

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = a_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - a_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + a_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

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$$\begin{vmatrix} 2 & 2 & 1 \\ 1 & -2 & 4 \\ 0 & 2 & -1 \end{vmatrix} = 2 \begin{vmatrix} -2 & 4 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix}$$

$$= -12 + 2 + 2 = -8.$$

Back to the cross product

In the language of determinants, we can write

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Example

Let $\vec{\mathbf{u}} = \langle 2, 1, 1 \rangle$ *and* $\vec{\mathbf{v}} = \langle -4, 3, 1 \rangle$. *Then we have*

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$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \vec{\mathbf{i}} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \vec{\mathbf{j}} \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + \vec{\mathbf{k}} \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$

$$= \vec{\mathbf{i}}(1-3) - \vec{\mathbf{j}}(2+4) + \vec{\mathbf{k}}(6+4) = -2\vec{\mathbf{i}} - 6\vec{\mathbf{j}} + 10\vec{\mathbf{k}}$$

Three equivalent formulas

The following formulas are all equally valid ways to find the cross product of two vectors.

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \sin(\theta))\vec{\mathbf{n}} = (u_2 v_3 - u_3 v_2)\vec{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\vec{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\vec{\mathbf{k}}$$

$$= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Properties of the cross product

The cross product satisfies several useful properties, which are given in the textbook at page 726.

1. $(r\vec{\mathbf{u}}) \times (s\vec{\mathbf{v}}) = (rs)(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$
2. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$
3. $\vec{\mathbf{v}} \times \vec{\mathbf{u}} = -(\vec{\mathbf{u}} \times \vec{\mathbf{v}})$
4. $(\vec{\mathbf{v}} + \vec{\mathbf{w}}) \times \vec{\mathbf{u}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}} + \vec{\mathbf{w}} \times \vec{\mathbf{u}}$
5. $\vec{\mathbf{0}} \times \vec{\mathbf{u}} = 0$
6. $\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{w}}$

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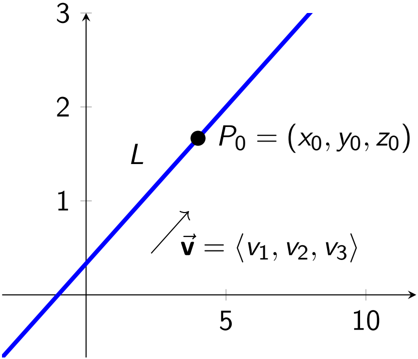
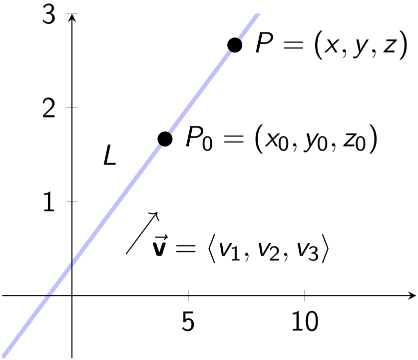


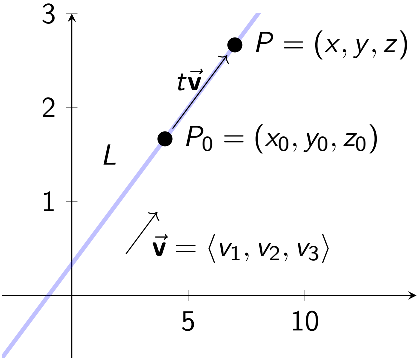
Lines

The point (x, y, z) is on the line L if and only if (x, y, z) is in the direction of $\vec{\mathbf{v}}$ from P_0.



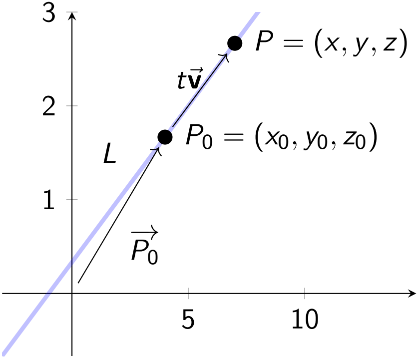
Lines

This means we can get from P_0 to P via some scalar multiple of $\vec{\mathbf{v}}$.



Lines

Thus, to write P as a vector (from the origin), we would first go to P_0 and then follow $t\vec{\mathbf{v}}$ to P.



So as vectors,

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle v_1, v_2, v_3\rangle.$$

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Definition

Let $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$ *and* $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ *be vectors. Then the vector equation of a line through* $P_0 = (x_0, y_0, z_0)$ *in the direction of* $\vec{\mathbf{v}}$ *is*

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}, \quad -\infty < t < \infty.$$

Lines

There are many equivalent ways to write the equation of a line.

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}} = \langle x_0, y_0, z_0 \rangle + t\langle v_1, v_2, v_3 \rangle = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle.$$

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Taking the last option, we can express a line as three parametric equations, where each variable is a function of t:

$$x(t) = x_0 + tv_1, \quad y(t) = y_0 + tv_2, \quad z(t) = z_0 + tv_3.$$

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Find the equation of the line L passing through the points $P_1 = (-3, 2, -3)$ *and* $P_2 = (1, -1, 4)$.

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If we wanted parametric equations for L, they would be $x = 1 + 4t, y = -1 - 3t, z = 4 + 7t$. The equations $x = -3 + 4s, y = 2 - 3s, z = -3 + 7s$ would also work.